Ramsey theory, Happy Ending problem

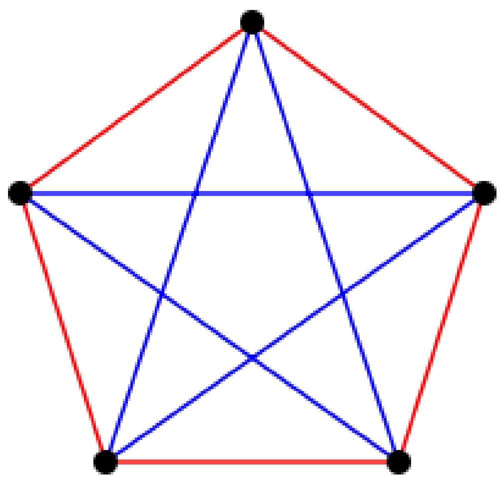
by Károly Zsolnai

Let's define friends as a group of people who all know each other, and strangers a group of people who don't. If you were to organize a party, how many people would you have to invite if you wanted to be sure that at least 3 friends or 3 strangers will attend?

or

If you open up a telephone book, how many people would you have to pick to be sure that at least 4 of them know each other or there are 2 of them who don't?

This is the main question of Ramsey theory. Mathematicians call these **Ramsey-numbers** and denote the first question as $\mathbf{R(3,3)}$, and the second $\mathbf{R(4,2)}$. Let's solve the first one by drawing a figure[1] to illustrate the scenario: the points (vertices) will represent people, and they will be connected with a red line if they are strangers, blue if they are friends. We call these drawings **graphs**.



1. Figure. Apparently, inviting just five people isn't enough.

As you can see, the above coloring is a good counterexample - look for a blue or red triangle! There is none, meaning it's possible to invite 5 people that no 3 of these people are friends or strangers. However, inviting 6 of them will always result in a good party (if we define a good party with what you can read above). Therefore $R(3,3) = 6$, and with a little work we can also conclude that if we choose 4 people from the telephone book, we'll also succeed, so $R(4,2) = 4$.

[1] source: http://en.wikipedia.org/wiki/Ramsey's_theorem

The theorem itself says there is a finite Ramsey-number for any scenario you can think of.

So, what's all this good for?

It's easy to understand this statement, though its implications are not quite obvious.

Imagine that there is an infinite number of worlds of any kind. Whatever they look like, there will be a common set of rules that will apply for all of them!

Isn't this *fascinating*? We can't say a lot, but would you have thought we can say anything at all?

This very important discovery led Hungarian mathematician **Klein Esther** to an interesting question: *if we draw 5 points onto a paper in any way, anywhere, can we always choose four of them so they will always give a convex shape?*
A shape is considered convex if it bulges outward - if you choose any two points in this shape and connect them with a ruler, the line will always be inside a shape. For instance, the red pentagon above is convex, and so is a circle, but a drawing of an hourglass is certainly not convex.

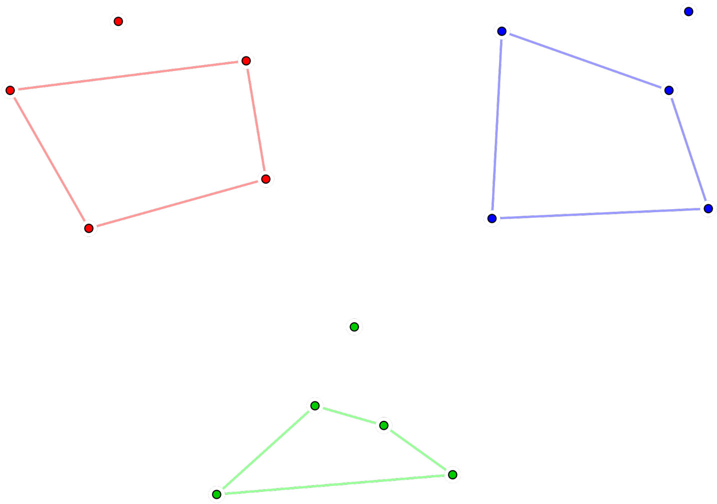
Probably not many of you would like to sit down and tackle the problem, and nor do I, to be honest! But I'm sure we agree that this is some kind of geometry problem. It's quite an interesting fact that the answer is **yes**, and it was formalised as a coloring problem, where the colors weren't indicating know/doesn't know or friend/stranger, but convex and non-convex (concave) relations and proved with Ramsey-theory! See the pictures on the next page[2]. George Szekeres and Esther Klein were working on this problem, which led to their marriage, hence Paul Erdős had given it the name Happy Ending problem. As of today, no one can solve it for arbitrary numbers.

What we have just discussed here can be summed up concisely in the language of mathematics:

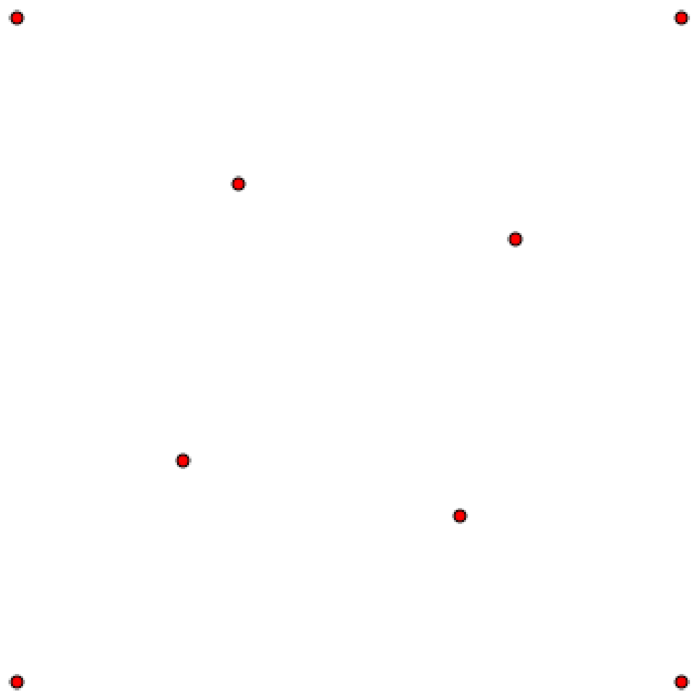
Any 2-coloring of a $K_{R(r,s)}$ graph will contain a monochromatic K_r or K_s. Let $R(r,s)$ be the smallest integer that satisfies this condition. $R(r,s) \in \mathbb{N}$ and is finite for any finite (r,s) combination.

And likewise, the main idea of the proving the Happy Ending problem is solving $R(4,2,n)$.

[2]source: http://en.wikipedia.org/wiki/Happy_Ending_problem



2. Figure. You can always choose 4 of them so connecting them will give a convex shape.



3. Figure. Choosing any five of these eight points will give a concave pentagon.