Teil 2: Kurven und Flächen

Parametrische Objekte

Kurven und Flächen

Kurven: 1D-Objekte

Flächen: 2D-Objekte,
basierend auf Kurven

Kurven

Welche Form der Darstellung?

Beispiel: 2D-Linie

- Explizit: $y = k \cdot x + d$ $\quad \mathbf{x} = (x, y)^T$
- Implzit: $a \cdot x + b \cdot y + c = 0$
  auch: $\mathbf{x} \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n}$ $\quad \mathbf{n} = (a, b)^T$
  $\mathbf{p} \cdot \mathbf{n} = -c$
- Parametrisch: $\mathbf{x}(t) = \mathbf{p} + t \cdot \mathbf{v}$ $\quad \mathbf{v} \cdot \mathbf{n} = 0$

Parametrische Kurven (1)

Definition:

- **x**(t) – Punktdef. in Abh. von kontinuierlichen Parameter t
- **x**(t) & t $\in$ [a,b] $\Rightarrow$ parametrische Kurve
- Startpunkt **x**(a), Endpunkt **x**(b)
- Beispiel: **x**(t) = **p** + t·**v** (Linie)

Tangente in x(t):

- $\dot{\mathbf{x}}(t) = d\mathbf{x}(t)/dt$ = Ableitung von **x** nach t
- Beispiel: $\dot{\mathbf{x}}(t) = \mathbf{v}$ (Tangentenvektor)

Parametrische Kurven (2)

2 Stetigkeitsqualitäten:

- mathematisch: $\dot{\mathbf{x}}(t)$ restriktiver
- geometrisch: $\dot{\mathbf{x}}(t) / |\dot{\mathbf{x}}(t)|$ schwächer

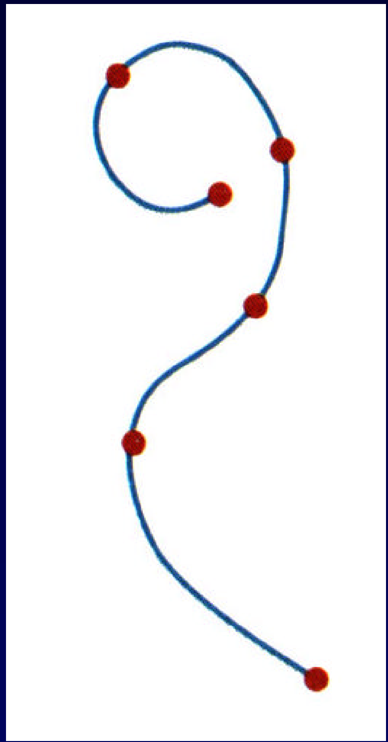
Attribute höherer Ordnung:

- Krümmung
- Torsion

Arten von Kurven

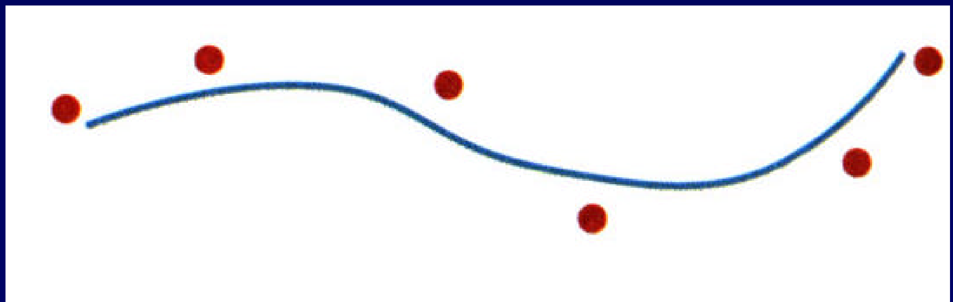
Interpolierende Kurven:

- Kurve geht durch vorgegebene Punkte $\mathbf{p}_i$



Approximierende Kurven:

- Kurve wird von Punkten $\mathbf{p}_i$ beeinflußt



Attribute von Kurven

Kurvenpunkte:

- Startpunkt, Endpunkt
- Kurvenevaluation

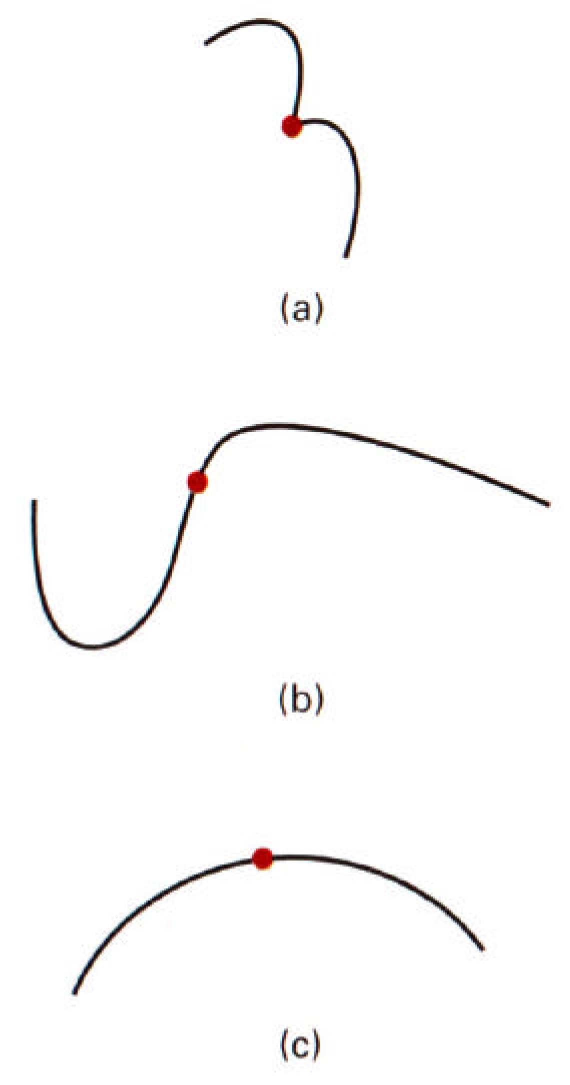
Tangenten:

- Startrichtung, Endrichtung
- Tangenten an beliebigen Kurvenpunkt

Stetigkeit: geometrisch, mathematisch

Grad: Differenzierbarkeit, smoothness

Spline-Kurven: Stetigkeiten



(a) C^0-Stetigkeit

(b) C^1-Stetigkeit

(c) C^2-Stetigkeit

Polynominterpolation

Gegeben: Anzahl von Punkten

Gesucht: Interpolierendes Polynom

- $p(x) = a + b \cdot x + c \cdot x^2 + \ldots + q \cdot x^n$
- 2 Punkte – Linie: $n = 1$
- 3 Punkte – Quadratisch Kurve: $n = 2$

Berechung:

- Lösung eines linearen Gleichungssystems

Beurteilung:

- Überschwingen (–), nur gut, wenn n klein

Spline-Kurven (1)

Kombination aus Teilkurven

Kontrollpunkte werden:

- approximiert
- interpoliert

Einfluß der Kontrollpunkte:

- global: Bézier-Kurve
- lokal: B-Spline-Kurve

Unterschiedlicher Grad

Spline-Kurven: Beispiele

2 Bsp. f. kubische Spline-Kurven

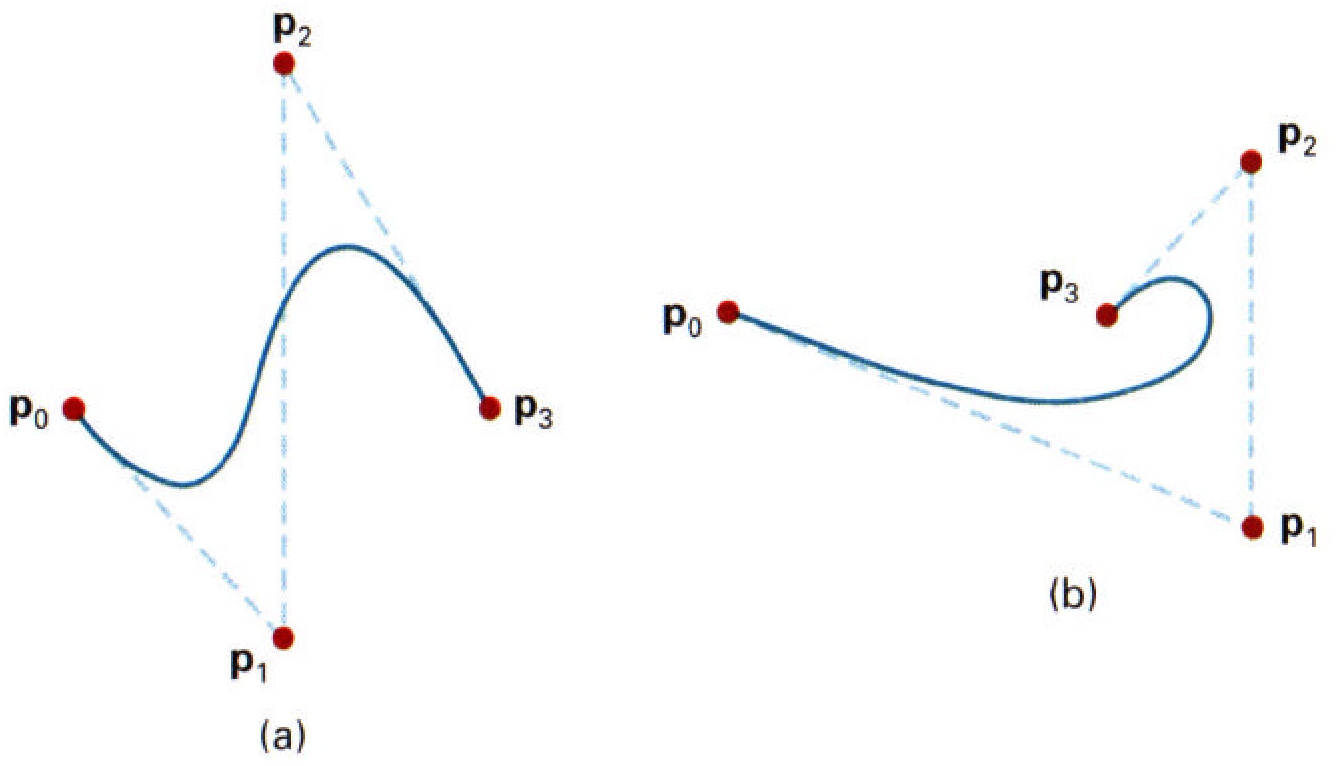


Figure 10-23
Control-graph shapes (dashed lines) for two different sets of control points.

Spline-Kurven (2)

Ansatz:

- Zusätzlich zu Kontrollpunkten $\mathbf{b}_i$ auch:
- Einflußfunktionen: Basis-Funktionen $B_{i,n}$

Definition:

- $\mathbf{b}(t) = \Sigma_{0 \le i \le n}\, \mathbf{b}_i \cdot B_{i,n}(t)$

Beispiel:

- Bernstein-Polynome (Bézier-Kurve)

Spline - convex hull property

Kurve bleibt i. d. konvexen Hülle

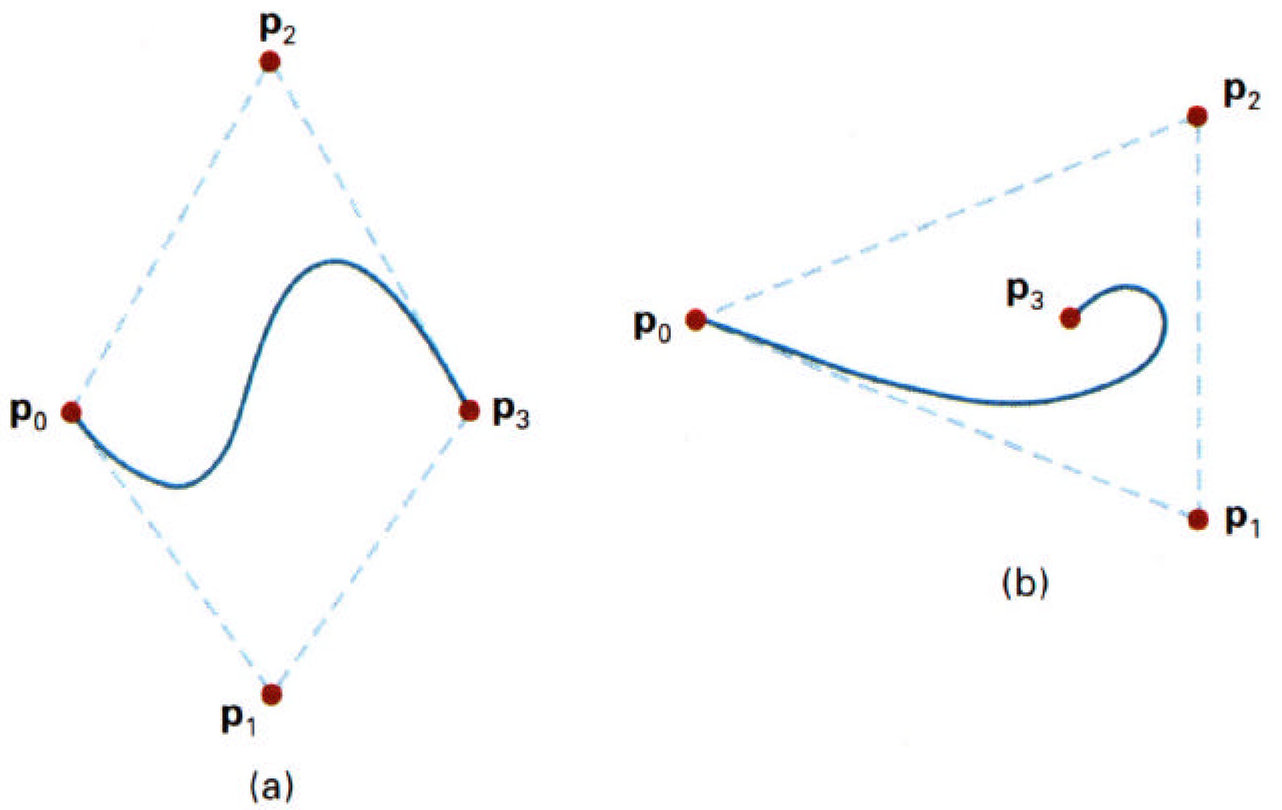


Figure 10-22
Convex-hull shapes (dashed lines) for two sets of control points.

Bézier-Kurven (1)

Spline-Approximation:

$$\mathbf{b}(t) = \sum_{i=0}^{n} \mathbf{b}_i \cdot B_{i,n}(t) \qquad 0 \leq t \leq 1$$

Bernstein Polynome:

$$B_{i,n}(t) = \begin{pmatrix} n \\ i \end{pmatrix} \cdot t^i \cdot (1-t)^{n-i}$$

Kubische Bézier-Kurven – Bernsteinpolynome

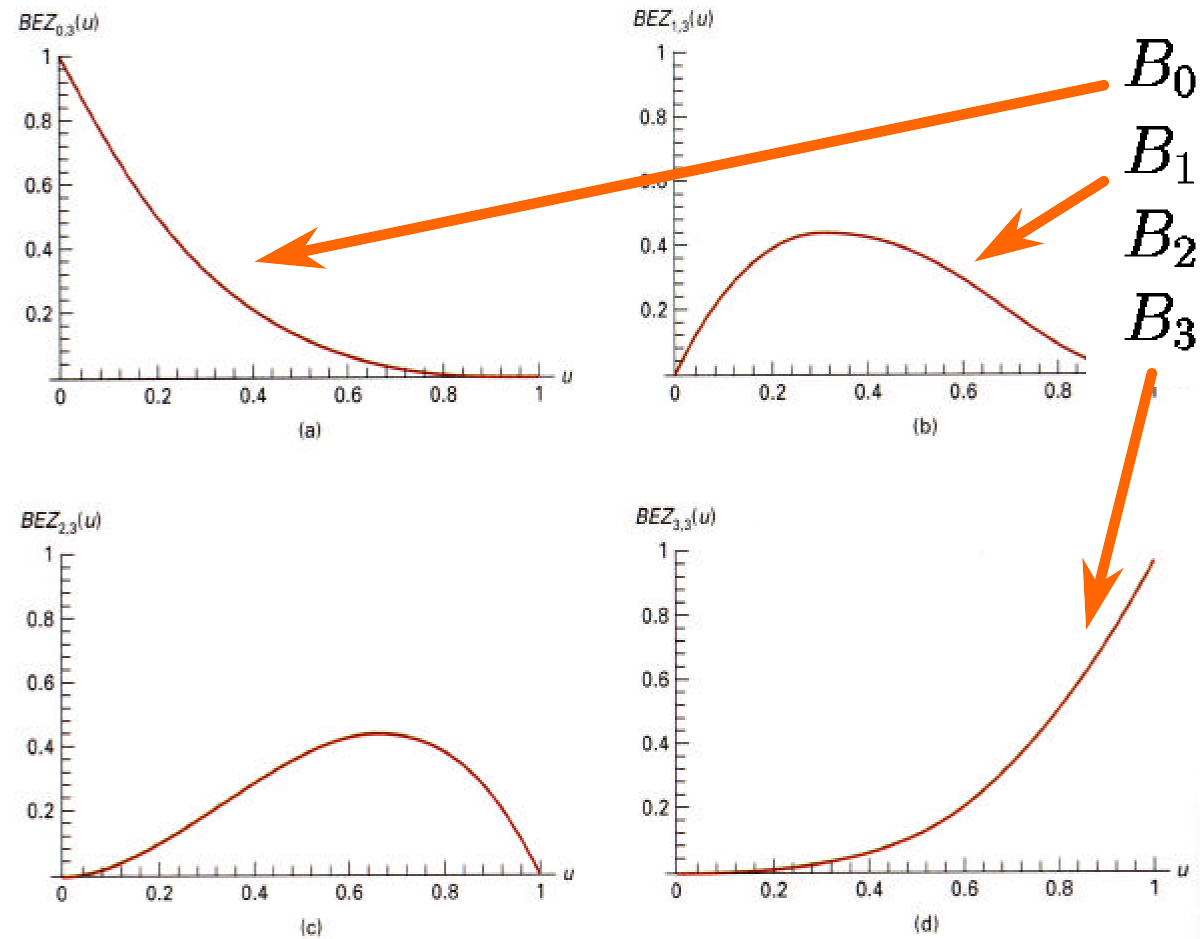


Figure 10-38
The four Bézier blending functions for cubic curves ($n = 3$).

$$
\begin{aligned}
B_{0,3}(t) &= (1-t)^3 \\
B_{1,3}(t) &= 3\,t \cdot (1-t)^2 \\
B_{2,3}(t) &= 3\,t^2 \cdot (1-t) \\
B_{3,3}(t) &= t^3
\end{aligned}
$$

Bézier-Kurven: Beispiel

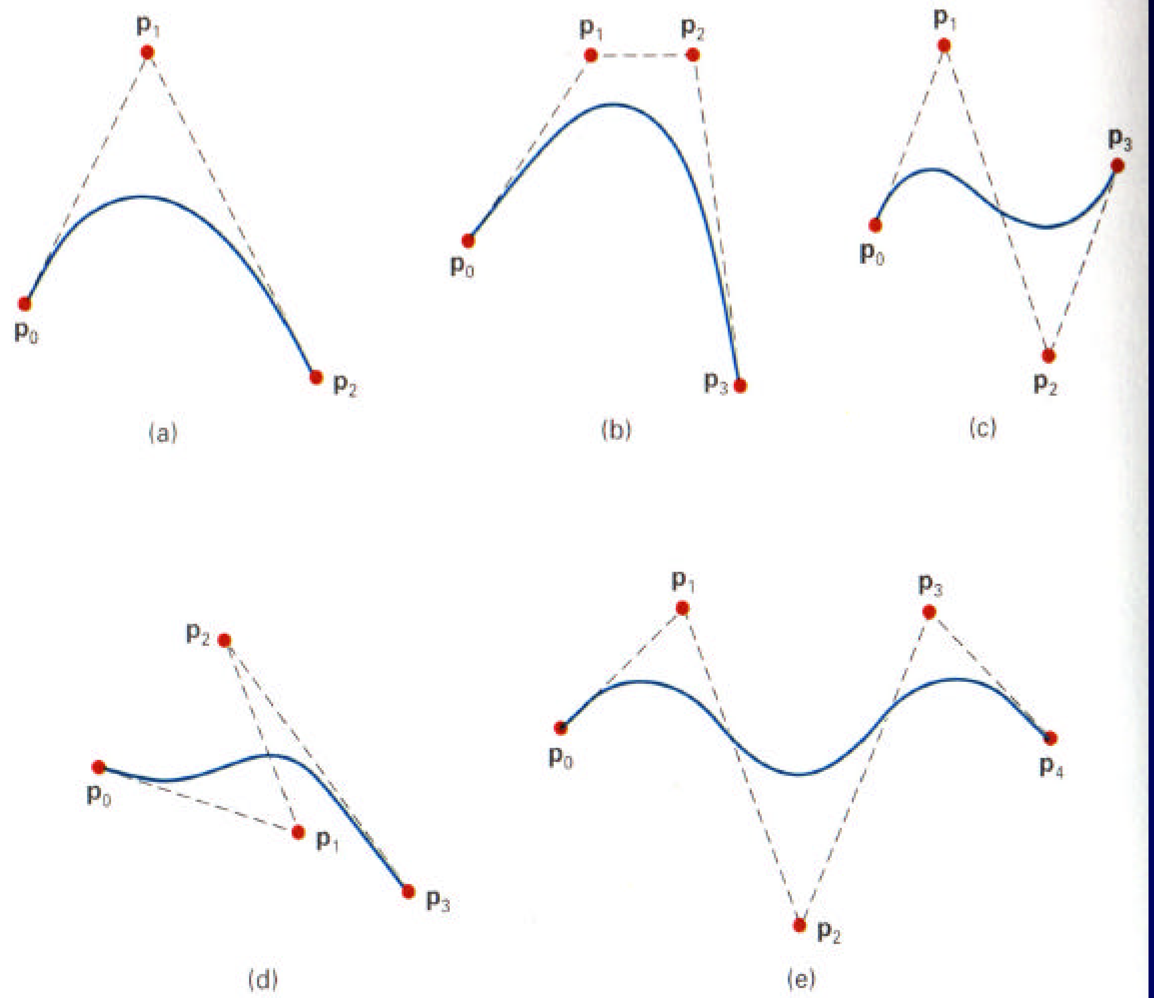


Figure 10-34
Examples of two-dimensional Bézier curves generated from three, four, and five control points. Dashed lines connect the control-point positions.

Bézier-Kurven (2)

Design mit Bézier-Kurven:

- closed Bézier curves
- multiple control points

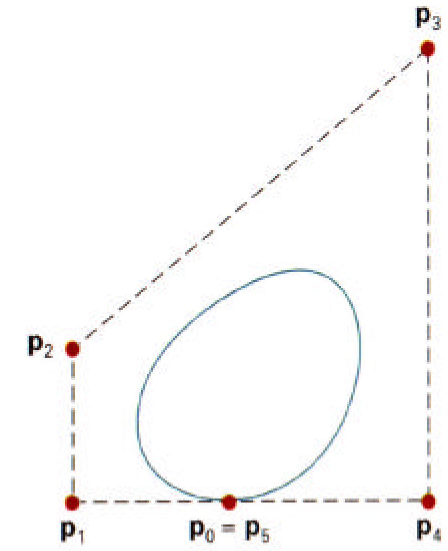


Figure 10-35
A closed Bézier curve generated by specifying the first and last control points at the same location.

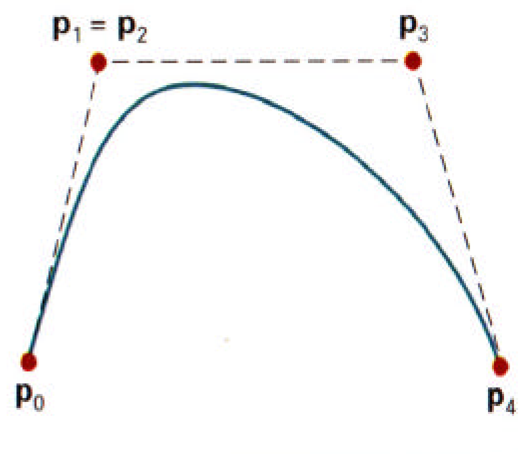


Figure 10-36
A Bézier curve can be made to pass closer to a given coordinate position by assigning multiple control points to that position.

Bézier-Kurven (3)

Design mit Bézier-Kurven

- Bézier verbinden (C^0, C^1 Stetigkeit)

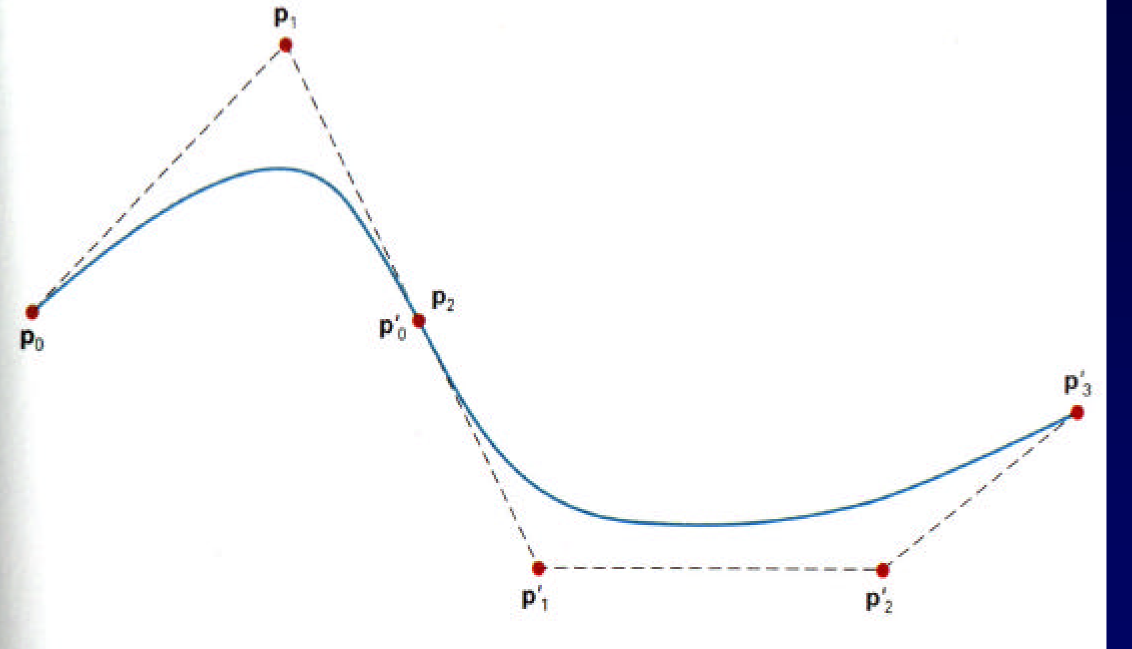


Figure 10-37
Piecewise approximation curve formed with two Bézier sections. Zero-order and first-order continuity are attained between curve sections by setting $\mathbf{p}'_0 = \mathbf{p}_2$ and by making points $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}'_1$ collinear.

Bézier-Kurve: Algorithmen

De Casteljau-Algorithmus:

- Evaluation bei Parameter t
- Rekursiver Ansatz

Degree elevation:

- Mehr Kontrollpunkte, selbe Kurve

Subdivision:

- Teilung der Kurve, bzw. Kurvenevaluation

Nächste Stufe: B-Spline Kurven

Bézier-Kurven: globaler Einfluß

Brauchbarer: B-Spline Kurven:

- Jeder Kontrollpunkt hat nur lokalen Einfluß
- Grad unabhängig von Anzahl der Pkte.

Rationale Kurven:

- Erweiterung von Bézier-Kurven und B-Spline Kurven
- Ein Gewicht pro Kontrollpunkt
- NURBS = „Standard“ in CAD, CAGD, etc.

Bézier-Patch (1)

Kartesisches Produkt v. Bézier-Kurven

$$\mathbf{b}(u,v) = \sum_{i=0}^{n} \sum_{j=0}^{n} \mathbf{b}_{i,j} \cdot B_{i,n}(u) \cdot B_{j,n}(v) \qquad 0 \le u, v \le 1$$

(m+1)x(n+1) Kontrollpunkte

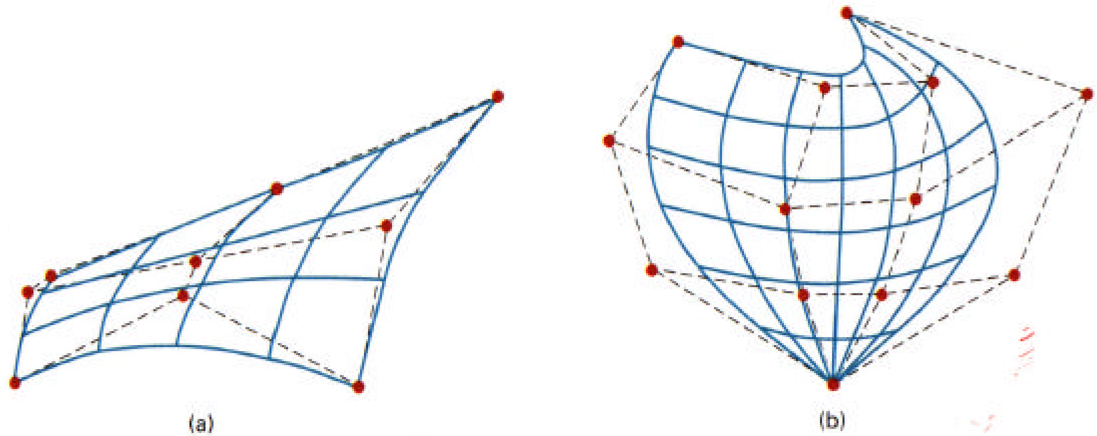


Figure 10-39
Bézier surfaces constructed for (a) $m = 3, n = 3$, and (b) $m = 4, n = 4$. Dashed lines connect the control points.

Bézier-Patch (2)

Eigenschaften wie Bézier-Kurve

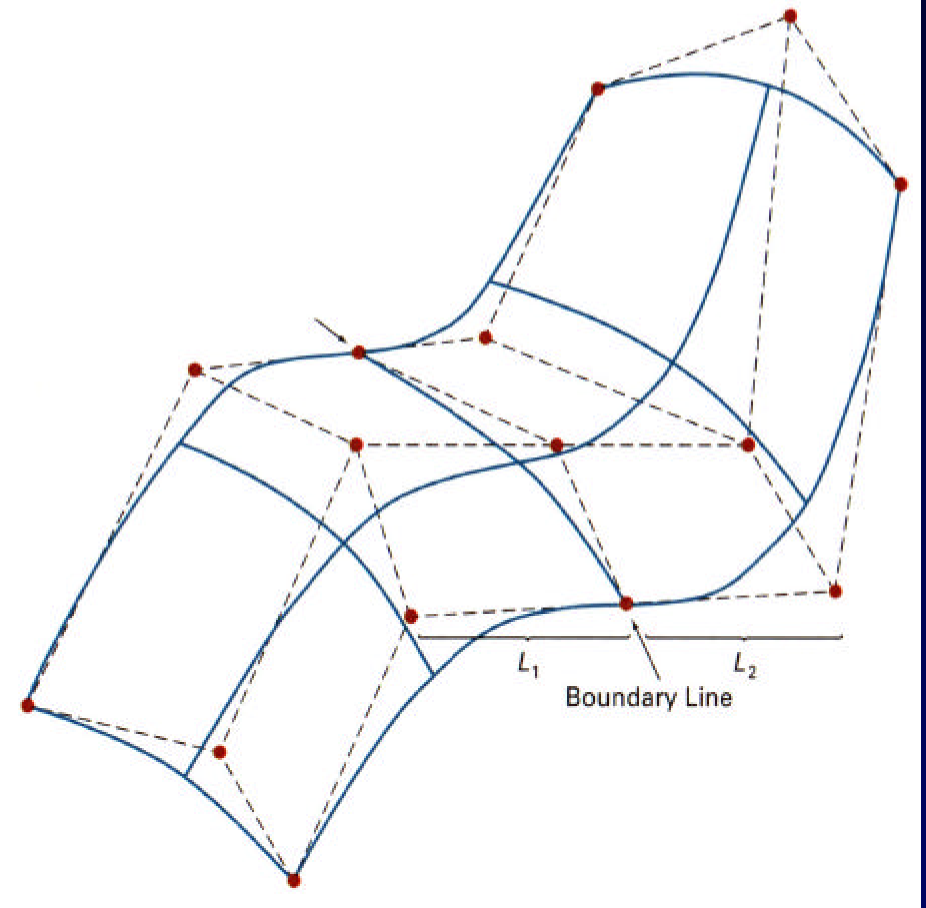


Figure 10-40
A composite Bézier surface constructed with two Bézier sections, joined at the indicated boundary line. The dashed lines connect specified control points. First-order continuity is established by making the ratio of length L_1 to length L_2 constant for each collinear line of control points across the boundary between the surface sections.

Quadric Surfaces (1)

Definition “quadrics”:

- Gleichung 2-ter Ordnung

Beispiel: Kugel

$$x^2 + y^2 + z^2 = r^2$$

$$\begin{aligned} x &= r \cdot \cos\phi \cdot \cos\theta & \pi \leq \theta \leq \pi \\ y &= r \cdot \cos\phi \cdot \sin\theta & \tfrac{\pi}{2} \leq \phi \leq \tfrac{\pi}{2} \\ z &= r \cdot \sin\phi & \end{aligned}$$

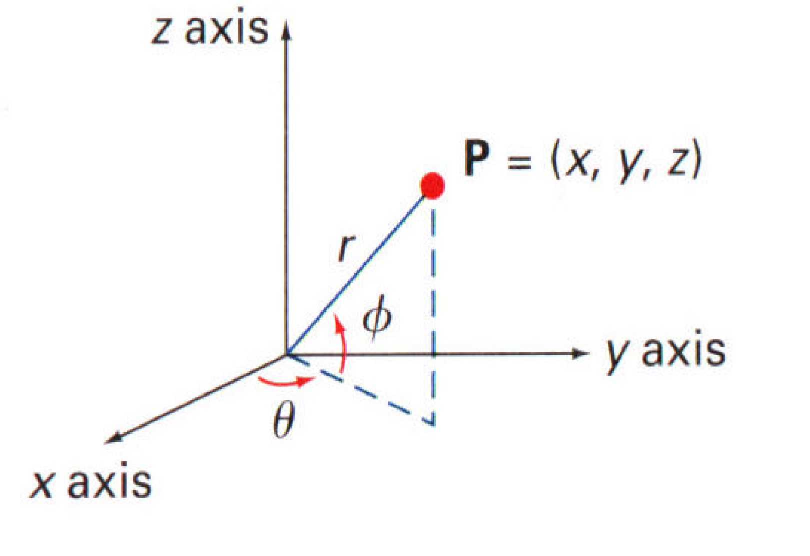


Figure 10-8
Parametric coordinate position (r, θ, ϕ) on the surface of a sphere with radius r.

Quadric Surfaces (2)

2. Beispiel: Ellipsoid

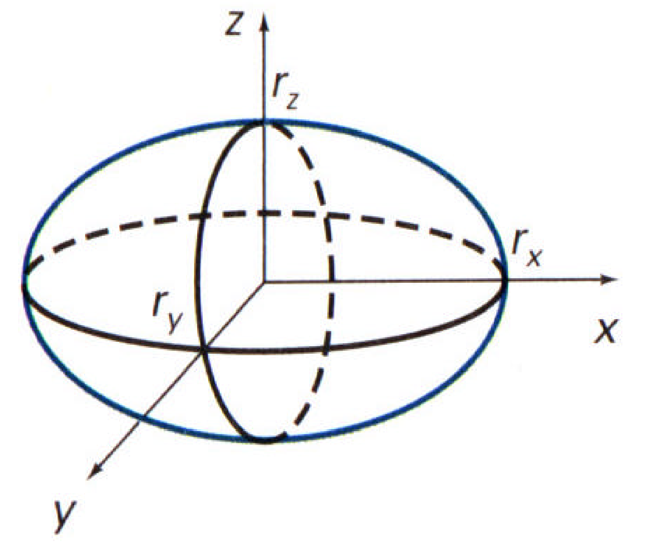


Figure 10-10
An ellipsoid with radii r_x, r_y, and r_z centered on the coordinate origin.

$$\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 = 1$$

$$\begin{aligned} x &= r_x \cdot \cos\phi \cdot \cos\theta \qquad && \pi \leq \theta \leq \pi \\ y &= r_y \cdot \cos\phi \cdot \sin\theta && \tfrac{\pi}{2} \leq \phi \leq \tfrac{\pi}{2} \\ z &= r_z \cdot \sin\phi && \end{aligned}$$

Quadric Surfaces (3)

3. Beispiel: Torus

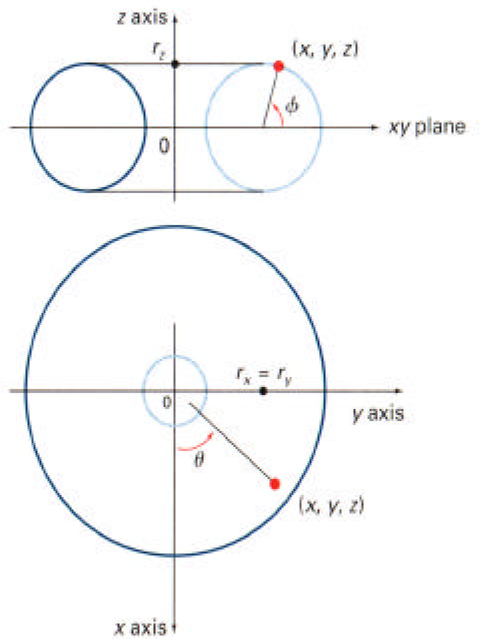


Figure 10-11
A torus with a circular cross section centered on the coordinate origin.

$$\left(r - \sqrt{\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2} \right)^2 + \left(\frac{z}{r_z}\right)^2 = 1$$

$$\begin{aligned} x &= r_x \cdot (r + \cos\phi) \cdot \cos\theta & \pi \le \theta \le \pi \\ y &= r_y \cdot (r + \cos\phi) \cdot \sin\theta & \pi \le \phi \le \pi \\ z &= r_z \cdot \sin\phi & \end{aligned}$$