

# Teil 2: Kurven und Flächen

## Parametrische Objekte

# Kurven und Flächen

**Kurven: 1D-Objekte**

**Flächen: 2D-Objekte,  
basierend auf Kurven**

# Kurven

## Welche Form der Darstellung?

### Beispiel: 2D-Linie

- ◆ Explizit:  $y = k \cdot x + d$        $\mathbf{x} = (x, y)^T$
- ◆ Implizit:  
auch:  $a \cdot x + b \cdot y + c = 0$   
 $\mathbf{x} \cdot \mathbf{n} = \mathbf{p} \cdot \mathbf{n}$        $\mathbf{n} = (a, b)^T$   
 $\mathbf{p} \cdot \mathbf{n} = -c$
- ◆ Parametrisch:  $\mathbf{x}(t) = \mathbf{p} + t \cdot \mathbf{v}$        $\mathbf{v} \cdot \mathbf{n} = 0$

# Parametrische Kurven (1)

## Definition:

- ◆  $\mathbf{x}(t)$  – Punktdef. in Abh. von kontinuierlichen Parameter  $t$
- ◆  $\mathbf{x}(t)$  &  $t \in [a, b] \Rightarrow$  parametrische Kurve
- ◆ Startpunkt  $\mathbf{x}(a)$ , Endpunkt  $\mathbf{x}(b)$
- ◆ Beispiel:  $\mathbf{x}(t) = \mathbf{p} + t \cdot \mathbf{v}$  (Linie)

## Tangente in $\mathbf{x}(t)$ :

- ◆  $\dot{\mathbf{x}}(t) = d\mathbf{x}(t)/dt =$  Ableitung von  $\mathbf{x}$  nach  $t$
- ◆ Beispiel:  $\dot{\mathbf{x}}(t) = \mathbf{v}$  (Tangentenvektor)

# Parametrische Kurven (2)

## 2 Stetigkeitsqualitäten:

- ◆ mathematisch:  $\dot{\mathbf{x}}(t)$  restriktiver
- ◆ geometrisch:  $\dot{\mathbf{x}}(t) / |\dot{\mathbf{x}}(t)|$  schwächer

## Attribute höherer Ordnung:

- ◆ Krümmung
- ◆ Torsion

# Arten von Kurven

## Interpolierende Kurven:

- ◆ Kurve geht durch vorgegebene Punkte  $p_i$

## Approximierende Kurven:

- ◆ Kurve wird von Punkten  $p_i$  beeinflußt



# Attribute von Kurven

## Kurvenpunkte:

- ◆ Startpunkt, Endpunkt
- ◆ Kurvenerzeugung

## Tangenten:

- ◆ Startrichtung, Endrichtung
- ◆ Tangenten an beliebigen Kurvenpunkt

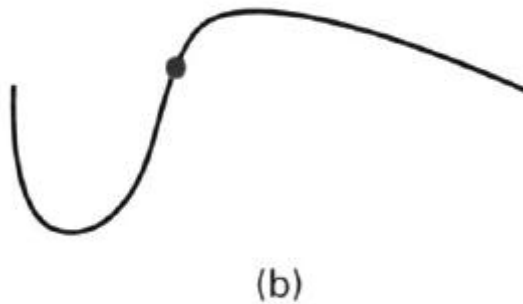
**Stetigkeit:** geometrisch, mathematisch

**Grad:** Differenzierbarkeit, smoothness

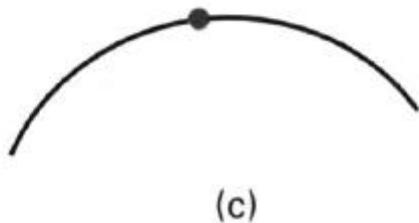
# Spline-Kurven: Stetigkeiten



**(a)  $C^0$ -Stetigkeit**



**(b)  $C^1$ -Stetigkeit**



**(c)  $C^2$ -Stetigkeit**



# Polynominterpolation

**Gegeben: Anzahl von Punkten**

**Gesucht: Interpolierendes Polynom**

◆  $p(x) = a + b \cdot x + c \cdot x^2 + \dots + q \cdot x^n$

◆ 2 Punkte – Linie:  $n = 1$

◆ 3 Punkte – Quadratisch Kurve:  $n = 2$

**Berechnung:**

◆ Lösung eines linearen Gleichungssystems

**Beurteilung:**

◆ Überschwingen (–), nur gut, wenn  $n$  klein

# Spline-Kurven (1)

**Kombination aus Teilkurven**

**Kontrollpunkte werden:**

- ◆ approximiert
- ◆ interpoliert

**Einfluß der Kontrollpunkte:**

- ◆ global: Bézier-Kurve
- ◆ lokal: B-Spline-Kurve

**Unterschiedlicher Grad**

# Spline-Kurven: Beispiele

## 2 Bsp. f. kubische Spline-Kurven

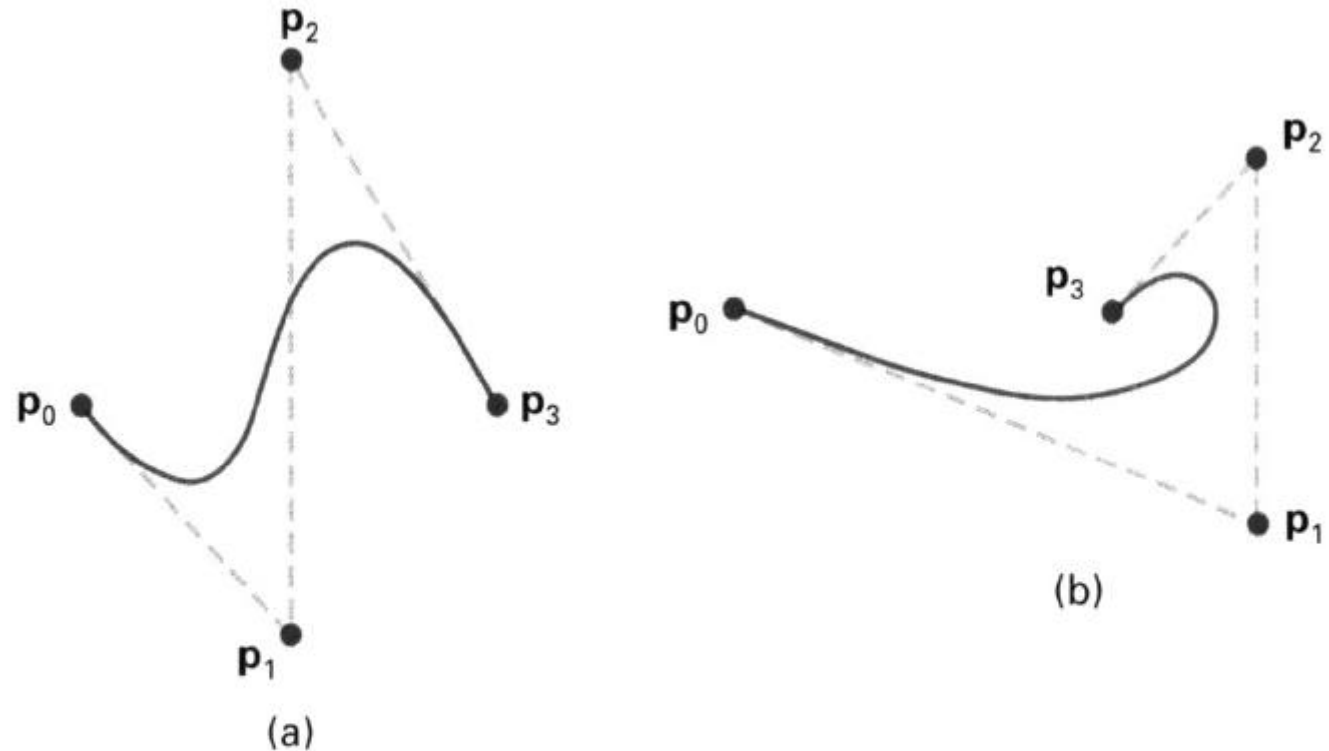


Figure 10-23

Control-graph shapes (dashed lines) for two different sets of control points.

# Spline-Kurven (2)

## Ansatz:

- ◆ Zusätzlich zu Kontrollpunkten  $\mathbf{b}_i$  auch:
- ◆ Einflußfunktionen: Basis-Funktionen  $B_{i,n}$

## Definition:

- ◆  $\mathbf{b}(t) = \sum_{0 \leq i \leq n} \mathbf{b}_i \cdot B_{i,n}(t)$

## Beispiel:

- ◆ Bernstein-Polynome (Bézier-Kurve)

# Spline - convex hull property

Kurve  
bleibt i. d.  
konvexen  
Hülle

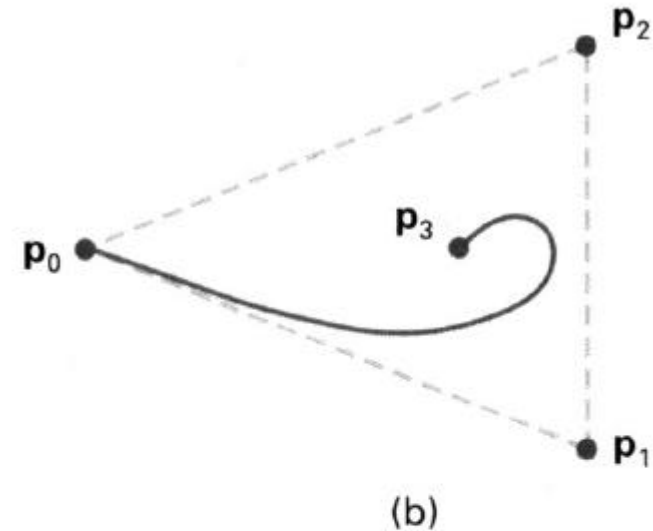
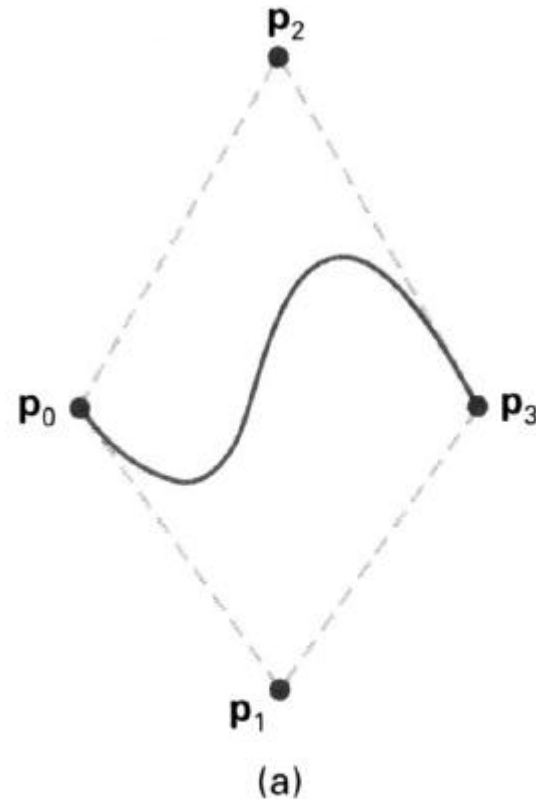


Figure 10-22

Convex-hull shapes (dashed lines) for two sets of control points.

# Bézier-Kurven (1)

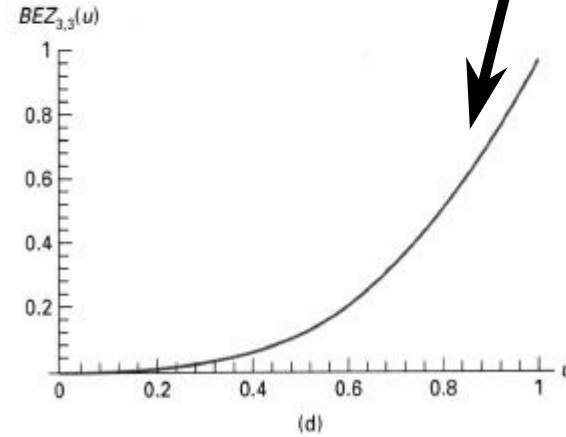
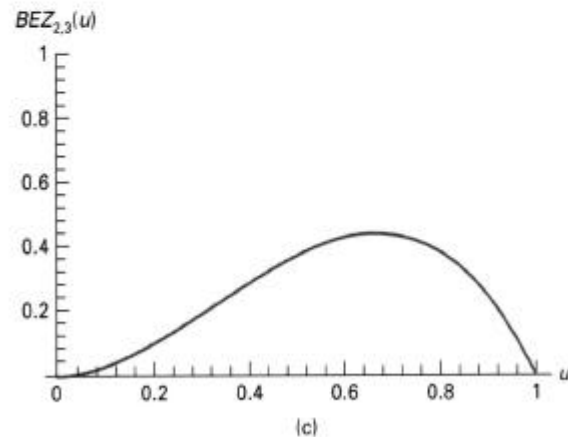
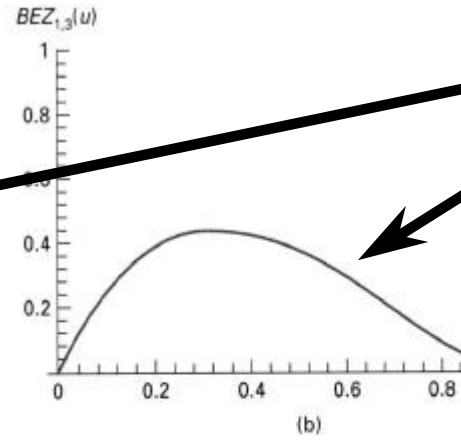
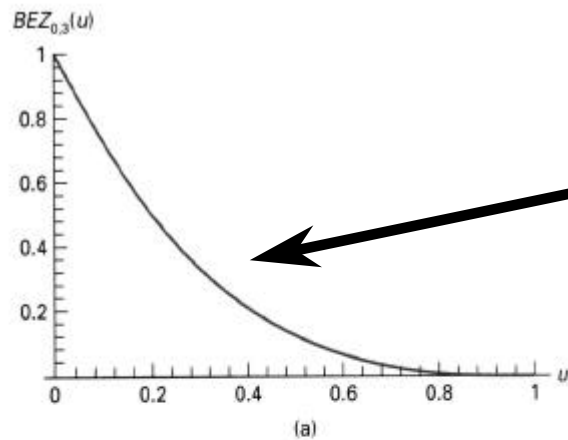
## Spline-Approximation:

$$\mathbf{b}(t) = \sum_{i=0}^n \mathbf{b}_i \cdot B_{i,n}(t) \quad 0 \leq t \leq 1$$

## Bernstein Polynome:

$$B_{i,n}(t) = \binom{n}{i} \cdot t^i \cdot (1-t)^{n-i}$$

# Kubische Bézier-Kurven – Bernsteinpolynome



$$\begin{aligned}
 B_{0,3}(t) &= (1-t)^3 \\
 B_{1,3}(t) &= 3t \cdot (1-t)^2 \\
 B_{2,3}(t) &= 3t^2 \cdot (1-t) \\
 B_{3,3}(t) &= t^3
 \end{aligned}$$

Figure 10-38  
The four Bézier blending functions for cubic curves ( $n = 3$ ).

# Bézier-Kurven: Beispiel

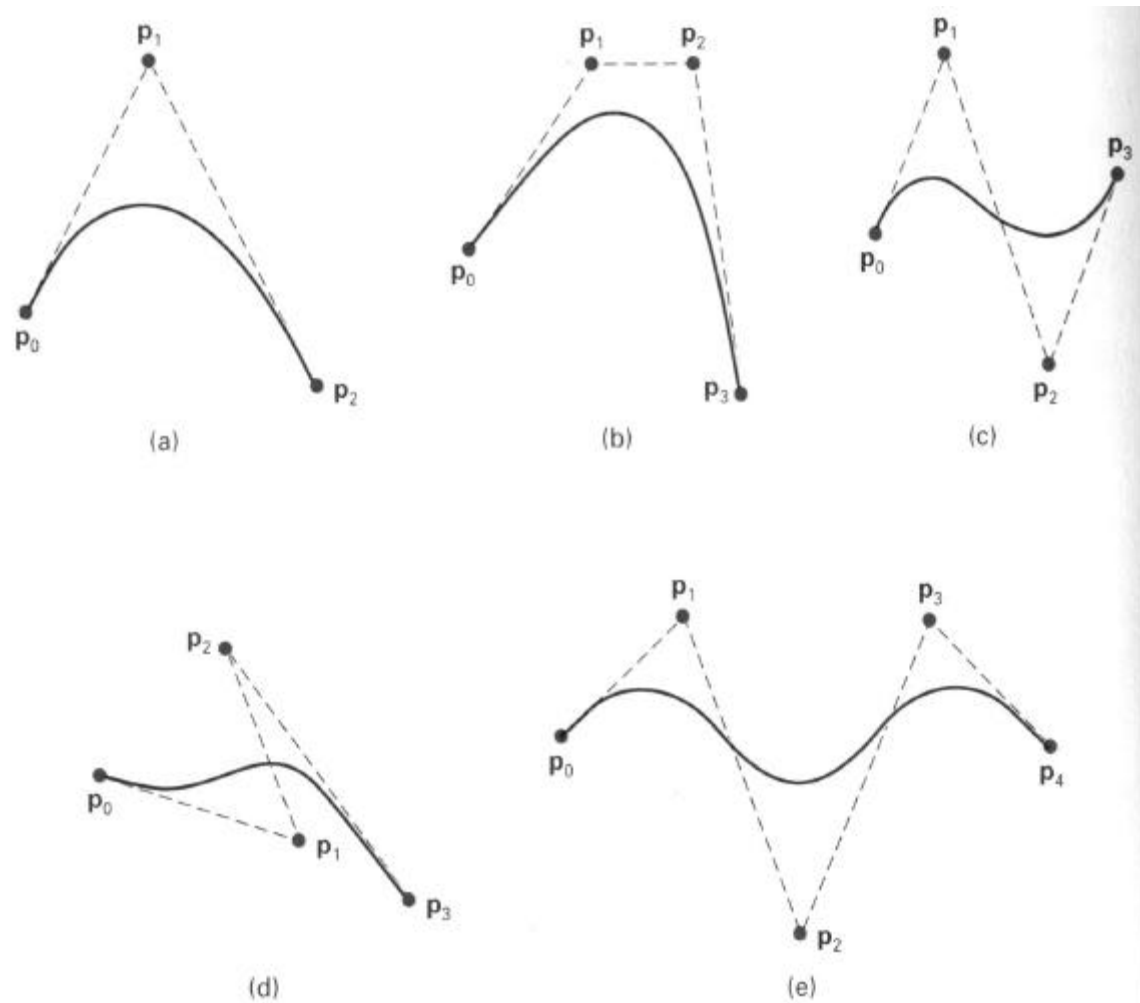


Figure 10-34

Examples of two-dimensional Bézier curves generated from three, four, and five control points. Dashed lines connect the control-point positions.



# Bézier-Kurven (2)

## Design mit Bézier-Kurven:

- ◆ closed Bézier curves
- ◆ multiple control points

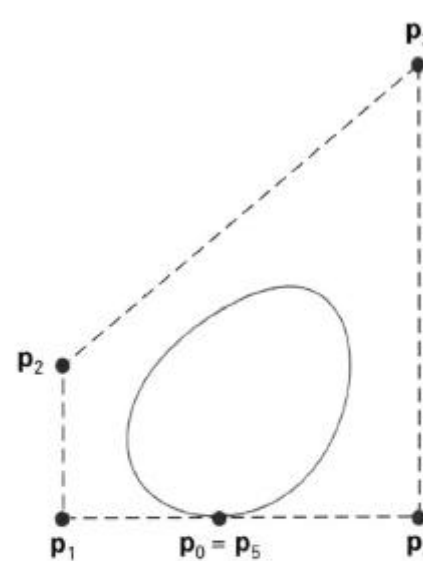


Figure 10-35  
A closed Bézier curve generated by specifying the first and last control points at the same location.

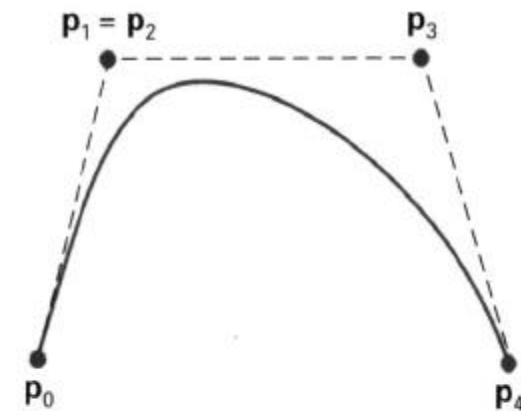


Figure 10-36  
A Bézier curve can be made to pass closer to a given coordinate position by assigning multiple control points to that position.

# Bézier-Kurven (3)

## Design mit Bézier-Kurven

- ◆ Bézier verbinden ( $C^0$ ,  $C^1$  Stetigkeit)

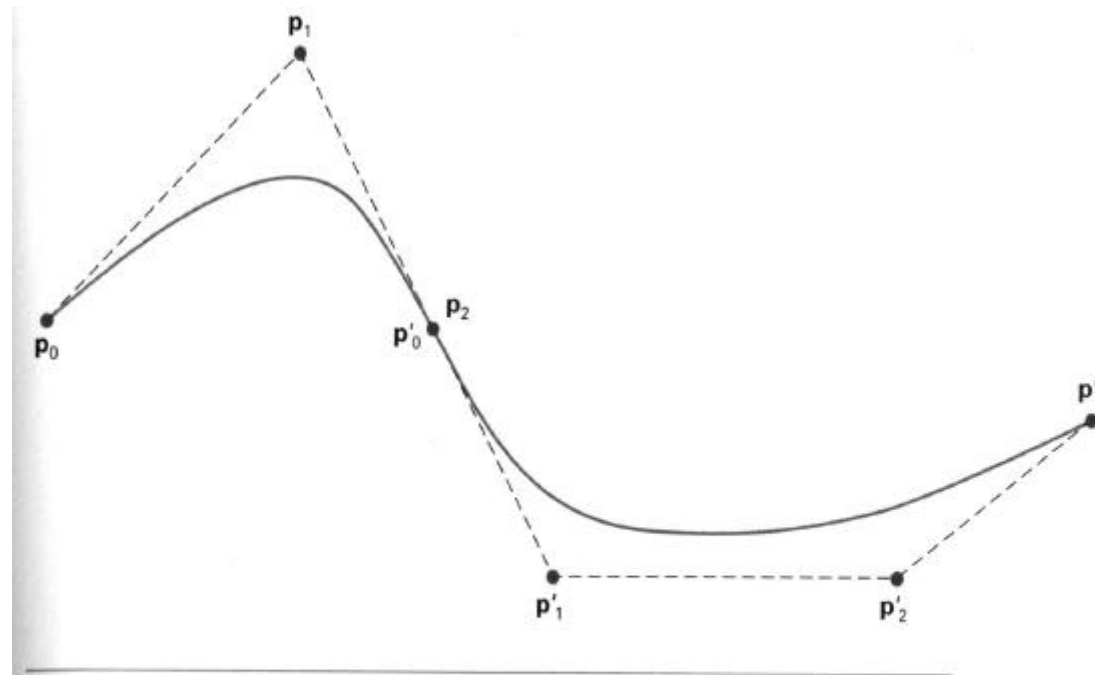


Figure 10-37

Piecewise approximation curve formed with two Bézier sections. Zero-order and first-order continuity are attained between curve sections by setting  $p'_0 = p_2$  and by making points  $p_1$ ,  $p_2$ , and  $p'_1$  collinear.

# Bézier-Kurve: Algorithmen

## De Casteljau-Algorithmus:

- ◆ Evaluation bei Parameter  $t$
- ◆ Rekursiver Ansatz

## Degree elevation:

- ◆ Mehr Kontrollpunkte, selbe Kurve

## Subdivision:

- ◆ Teilung der Kurve, bzw. Kurvenevaluation

# Nächste Stufe: B-Spline Kurven

**Bézier-Kurven: globaler Einfluß**

**Brauchbarer: B-Spline Kurven:**

- ◆ Jeder Kontrollpunkt hat nur lokalen Einfluß
- ◆ Grad unabhängig von Anzahl der Pkte.

**Rationale Kurven:**

- ◆ Erweiterung von Bézier-Kurven und B-Spline Kurven
- ◆ Ein Gewicht pro Kontrollpunkt
- ◆ NURBS = „Standard“ in CAD, CAGD, etc.

# Bézier-Patch (1)

## Kartesisches Produkt v. Bézier-Kurven

$$\mathbf{b}(u, v) = \sum_{i=0}^n \sum_{j=0}^n \mathbf{b}_{i,j} \cdot B_{i,n}(u) \cdot B_{j,n}(v) \quad 0 \leq u, v \leq 1$$

**(m+1)x(n+1) Kontrollpunkte**

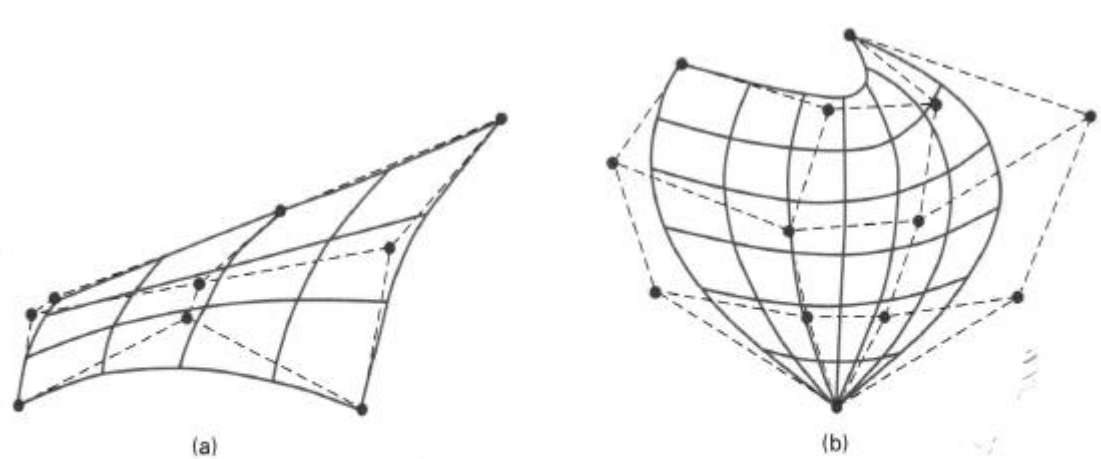


Figure 10-39

Bézier surfaces constructed for (a)  $m = 3, n = 3$ , and (b)  $m = 4, n = 4$ . Dashed lines connect the control points.

# Bézier-Patch (2)

## Eigenschaften wie Bézier-Kurve

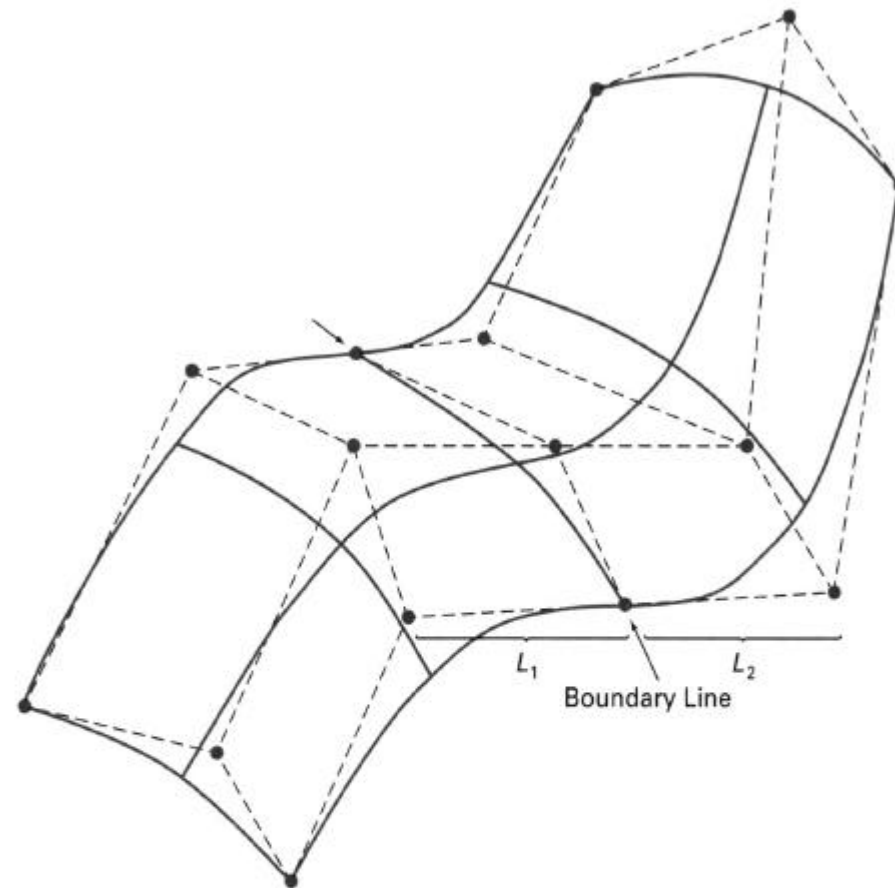


Figure 10-40

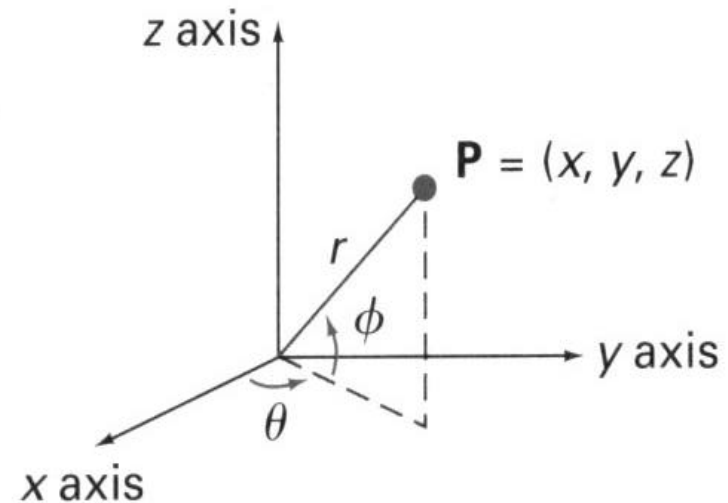
A composite Bézier surface constructed with two Bézier sections, joined at the indicated boundary line. The dashed lines connect specified control points. First-order continuity is established by making the ratio of length  $L_1$  to length  $L_2$  constant for each collinear line of control points across the boundary between the surface sections.

# Quadric Surfaces (1)

## Definition “quadrics”:

- ◆ Gleichung 2-ter Ordnung

## Beispiel: Kugel



$$x^2 + y^2 + z^2 = r^2$$

$$x = r \cdot \cos \phi \cdot \cos \theta$$

$$y = r \cdot \cos \phi \cdot \sin \theta$$

$$z = r \cdot \sin \phi$$

$$\pi \leq \theta \leq \pi$$

$$\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

Figure 10-8

Parametric coordinate position  $(r, \theta, \phi)$  on the surface of a sphere with radius  $r$ .

# Quadric Surfaces (2)

## 2. Beispiel: Ellipsoid

$$\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2 + \left(\frac{z}{r_z}\right)^2 = 1$$

$$x = r_x \cdot \cos \phi \cdot \cos \theta \quad \pi \leq \theta \leq \pi$$

$$y = r_y \cdot \cos \phi \cdot \sin \theta \quad \frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}$$

$$z = r_z \cdot \sin \phi$$

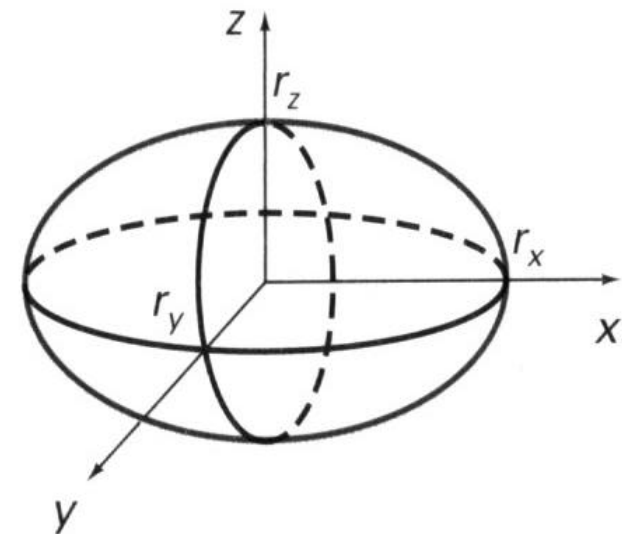


Figure 10-10

An ellipsoid with radii  $r_x$ ,  $r_y$ , and  $r_z$  centered on the coordinate origin.



# Quadric Surfaces (3)

## 3. Beispiel: Torus

$$\left( r - \sqrt{\left(\frac{x}{r_x}\right)^2 + \left(\frac{y}{r_y}\right)^2} \right)^2 + \left(\frac{z}{r_z}\right)^2 = 1$$

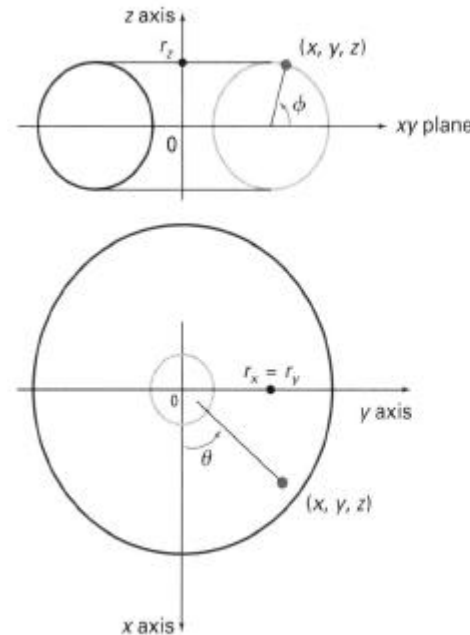


Figure 10-11  
A torus with a circular cross section centered on the coordinate origin.

$$\begin{aligned} x &= r_x \cdot (r + \cos \phi) \cdot \cos \theta & \pi &\leq \theta \leq \pi \\ y &= r_y \cdot (r + \cos \phi) \cdot \sin \theta & \pi &\leq \phi \leq \pi \\ z &= r_z \cdot \sin \phi \end{aligned}$$